INDIAN STATISTICAL INSTITUTE, CHENNAI CENTRE

M. STAT.(NB-STREAM)-I YEAR

ANALYSIS-I

END SEMESTER EXAMINATION-NOVEMBER 2016

Time: 3 Hours **Marks : 50**

Answer all the questions

1. State the following (without proof): $(3 \times 1 = 3$ Marks)
   (a) Heine-Borel theorem on $\mathbb{R}$.
   (b) Second fundamental theorem of calculus.
   (c) Weirstrass approximation theorem.

2. (a) Let X be the set of all convergent sequences in $\mathbb{R}$. Define $d(x, y) = \sup_i |x_i - y_i|$, for $x, y \in X$. Show that (X, d) is a metric space. (2 Marks)
   (b) Let $\mathbb{Z}[x]$ denote the set of all polynomials with integer coefficients. Let
   $$A := \{a \in \mathbb{R} \mid \text{there exists } p \in \mathbb{Z}[x] \text{ such that } p(a) = 0\}.$$
   Is A closed? Is A open? Justify your answer. (2 Marks)
   (c) Show that a subset K of $\mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a point in K. (2 Marks)

3. (a) Give an example of a function f on $\mathbb{R}$ which is continuous at every irrational point and not continuous at any rational point of $\mathbb{R}$. Justify your answer. (2 Marks)
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $x = 0$. Let $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$, and $f(x) = \lambda x$, where $\lambda = f(1)$, for $x \in \mathbb{R}$. (4 Marks)
   (c) Define the types of discontinuities of a real valued function defined on an interval. Show that if f is monotonic on (a, b), then the set of points of (a, b) at which f is discontinuous is at most countable. (5 Marks)

4. (a) Give an example of a function f that is differentiable at every point of $\mathbb{R}$ and does not have a continuous derivative f'. Justify your answer. (2 Marks)
   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$, and $f(b) = b$. Show that there exists $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there exist $c_1, c_2 \in (a, b)$ such that $c_1 \neq c_2$ and $f'(c_1) + f'(c_2) = 2$. (2 Marks)
   (c) Suppose that f is differentiable on $\mathbb{R}$ and such that $|f'(x)| \leq \lambda < 1$, for all $x \in \mathbb{R}$. Show that $f(x) = x$ has a unique solution. (3 Marks)
   (d) State and prove the Taylor's theorem with Lagrange's form of remainder. (4 Marks)

5. (a) Let f be a real valued bounded function on $[a, b]$, and let f be monotonically increasing on $[a, b]$. Show that f is integrable on $[a, b]$. (3 Marks)
   (b) If f is integrable on $[a, b]$, then show that $|f|$ is integrable on $[a, b]$, and $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$.
   Hence evaluate $\lim_{n\to\infty} \int_0^{2\pi} \frac{\sin nx}{x^2 + n^2} dx$. (5 Marks)
   (c) Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos\left(\frac{2k-1}{2n}\right)$ using Riemann integrals. (2 Marks)

6. (a) State and prove the Weirstrass M-test for the series of functions. (3 Marks)
   (b) Assume that $f_n \to f$ uniformly on $[a, b]$, and that each f_n is integrable. Show that f is integrable, and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$. Show by an example that the hypothesis $f_n \to f$ uniformly on $[a, b]$ can not be replaced by $f_n \to f$ pointwise on $[a, b]$. (4 + 2 Marks)

ALL THE BEST